On Path diagrams and Stirling permutations

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Abstract

Any ordinary permutation $\tau \in S_n$ of size n, written as a word $\tau = \tau_1 \dots \tau_n$, can be locally classified according to the relative order of τ_j to its neighbours. This gives rise to four local order types called peaks (or maxima), valleys (or minima), double rises and double falls. By the correspondence between permutations and binary increasing trees the classification of permutations according to local types corresponds to a classification of binary increasing trees according to nodes types. Moreover, by the bijection between permutations, binary increasing trees and suitably defined path diagrams one can obtain continued fraction representations of the ordinary generating function of local types. The aim of this work is to introduce the notion of local types in k-Stirling permutations, to relate these local types with nodes types in $(k+1)$-ary increasing trees and to obtain a bijection with suitably defined path diagrams. Furthermore, we also present a path diagram representation of a related tree family called plane-oriented recursive trees, and the discuss the relation with ternary increasing trees.

Keywords: Path diagrams, Stirling permutations, Increasing trees, local types, formal power series
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1 Introduction

Any ordinary permutation $\tau = \tau_1 \dots \tau_n$ of size n can be locally be classified according to four local types called peaks (maxima), valleys (minima), double rises and double falls, depending on the relative order of τ_j to its neighbours. Index j is called a peak if $\tau_{j-1} < \tau_j > \tau_{j+1}$, a valley if $\tau_{j-1} > \tau_j < \tau_{j+1}$, a double rise if $\tau_{j-1} < \tau_j < \tau_{j+1}$, and a double fall if $\tau_{j-1} > \tau_j > \tau_{j+1}$; for $1 \le j \le n$, with respect to the boundary conditions $\tau_0 = \tau_{n+1} = -\infty$, see Flajolet [7], or Conrad and Flajolet [5] and the references therein. Moreover, due to the bijection with binary increasing trees [7, 1], there exists a correspondence to certain local node types in binary increasing trees [7, 5]. Flajolet [7], see also Françon and Viennot [9], used a path diagrams representation of permutations or equivalently binary increasing trees to obtain a continued fraction representation of the ordinary generating function of local types in permutations, or equivalently of node types in binary increasing trees.

Stirling permutations were defined by Gessel and Stanley [10]. A Stirling permutation $\sigma = \sigma_1 \dots \sigma_{2n}$ is a permutation of the multiset $\{1, 1, 2, 2, \dots, n, n\}$ such that for each i, $1 \le i \le n$, the elements occurring between the two occurrences of i are larger than i. The name of these combinatorial objects is due to relations with the Stirling numbers, see [10] for details. Recently, this class of combinatorial objects have generated some interest. Bona [2] studied the distribution of descents in Stirling permutation.

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Janson [14] showed the connection between Stirling permutations and plane-oriented recursive trees and proved a joint normal limit law for the parameters considered by Bona.

A natural generalization of Stirling permutations on the multiset $\{1,1,\dots,n,n\}$ is to consider permutations of a more general multiset $\{1^k,2^k,\dots,n^k\}$, with $k\in\mathbb{N}=\{1,2,\dots\}$. We call a permutation of the multiset $\{1^k,2^k,\dots,n^k\}$ a *k-Stirling permutation*, if for each i, $1\le i\le n$, the elements occurring between two occurrences of i are at least i. Such generalized Stirling permutations have already previously been considered by Brenti [3], [4], and also by Park [19, 20, 21], albeit under the different name k-multipermutations. For $k=1$ one obtains ordinary permutations. For $k=2$ the class of 2-Stirling permutations coincides with the ordinary Stirling permutations introduced by Gessel and Stanley. Recently, Janson et al. [15] studied several parameters in k-Stirling permutations, related to the studies [2, 14]: they extended the results of [2, 14] concerning the distribution of descents and related statistics. An important result of [15] is the natural bijection between k-Stirling permutations and $(k+1)$-ary increasing trees, for $k\ge 1$, which was already known to Gessel (see Park [19]). The family of $(k+1)$-ary increasing trees, with integer $k\in\mathbb{N}$, includes the well known family of binary increasing trees $k=1$, and also the family of ternary increasing trees, $k=2$.

The aims of this work are threefold. First, we introduce the notion of local types in k-Stirling permutations and also of local node types in the corresponding $(k+1)$-ary increasing trees. Second, we give a bijection between k-Stirling permutations, $(k+1)$-ary increasing trees and suitably defined path diagrams. Third, we use the relation between path diagrams and formal power series to obtain a continued fraction type expansion of the generating function of local types in k-Stirling permutations, or equivalently local node types in $(k+1)$-ary increasing trees. Furthermore, we also discuss a related tree family called plane-oriented recursive trees,[1] which is in bijection with 2-Stirling permutations [14], and also with ternary increasing trees [15]. We obtain a path diagram representation of plane-oriented recursive trees and discuss the implication of the bijection with ternary increasing trees, as given in [15]. Best to the authors knowledge these problems have not been addressed before in the literature. This study is motivated by the work [7], partly also by [5], and by Gessel's bijection between k-Stirling permutations and $(k+1)$-ary increasing trees [15], see also [19].

2 Increasing trees and generalized Stirling permutations

2.1 Generalized Stirling permutations

Let $\mathcal{Q}_n=\mathcal{Q}_n(k)$ denote the set of k-Stirling permutations of size n and let $Q_n=Q_n(k)$ denote the number $|\mathcal{Q}_n(k)|$ of them. The number $|\mathcal{Q}_n(k)|$ is given by

$$Q_n=|\mathcal{Q}_n|=\prod_{i=1}^{n-1}(ki+1)=k^n\frac{\Gamma(n+1/k)}{\Gamma(1/k)}, \tag{1}$$

since the k copies of n have to form a substring, and this substring can be inserted in $k(n-1)+1$ positions, anywhere–including first or last position, in any k-Stirling permutation of size $n-1$; see for example [19, 15]. For $k=2$ this number is just $Q_n(2)=\prod_{i=1}^{n-1}(2i+1)=(2n-1)!!$. For example, in the case $k=3$ we have one permutation of size 1 given by 111; four permutations of size 2 given by 111222, 112221, 122211, 222111; etc. In order to relate the k-Stirling permutations to $(k+1)$-ary increasing trees, and to relate 2-Stirling permutations to plane-oriented recursive trees, we will introduce a general family of increasing trees. We use a setting based on earlier considerations of Bergeron et al. [1] and Panholzer and Prodinger [18], which also includes $(k+1)$-ary increasing trees an plane-oriented recursive trees as special instances. Although the tree families and their combinatorial

[1] The family of plane-oriented recursive trees also appears in the literature under the names plane recursive trees [14], heap-ordered trees [22, 23], Scale-free trees, and Barabási-Albert trees; see [11].

description is quite well known, we collect the most important considerations of [1, 18, 15] for the readers convenience.

2.2 Families of Increasing trees

Increasing trees are labeled trees, where the nodes of a tree of size n are labeled by distinct integers of the set $\{1, \dots, n\}$ in such a way that each sequence of labels along any branch starting at the root is increasing. As the underlying (unlabeled) tree model ones uses the so-called simply generated trees [17] but, additionally, the trees are equipped with increasing labellings. Thus, we are considering simple families of increasing trees, which are introduced in [1].

Formally, a class $\mathcal{T}$ of a simple family of increasing trees can be defined in the following way. A sequence of non-negative numbers $(\varphi_\ell)_{\ell\geq 0}$ with $\varphi_0 > 0$ called the degree-weight sequence, we further assume that there exists a $\ell \geq 2$ with $\varphi_\ell > 0$, is used to define the weight $w(T)$ of any ordered tree T by $w(T) := \prod_v \varphi_{\deg^+(v)}$, where v ranges over all vertices of T and $\deg^+(v)$ is the out-degree of v. Furthermore, $\mathcal{L}(T)$ denotes the set of different increasing labellings of the tree T with distinct integers $\{1, 2, \dots, |T|\}$, where $|T|$ denotes the size of the tree T, and $L(T) := \big|\mathcal{L}(T)\big|$ its cardinality. Then the family $\mathcal{T}$ consists of all trees T together with their weights $w(T)$ and the set of increasing labellings $\mathcal{L}(T)$. The simple family of increasing trees $\mathcal{T}$ associated with a degree-weight generating function $\varphi(t)$, can be described by the formal recursive equation

$$\mathcal{T} = ① \times \Big(\varphi_0 \cdot \{\epsilon\} \,\dot\cup\, \varphi_1 \cdot \mathcal{T} \,\dot\cup\, \varphi_2 \cdot \mathcal{T} * \mathcal{T} \,\dot\cup\, \varphi_3 \cdot \mathcal{T} * \mathcal{T} * \mathcal{T} \,\dot\cup\, \cdots\Big) = ① \times \varphi(\mathcal{T}), \qquad (2)$$

where ① denotes the node labeled by 1, $\times$ the cartesian product, $\dot\cup$ the disjoint union, $*$ the partition product for labeled objects, and $\varphi(\mathcal{T})$ the substituted structure; see e. g., the books [24, 8]. For a given degree-weight sequence $(\varphi_\ell)_{\ell\geq 0}$ with a degree-weight generating function $\varphi(t) := \sum_{\ell\geq 0}\varphi_\ell t^\ell$, we define now the total weights by $T_n := \sum_{|T|=n} w(T)\cdot L(T)$. It follows then that the exponential generating function $T(z) := \sum_{n\geq 1} T_n \frac{z^n}{n!}$ satisfies the autonomous first order differential equation

$$T'(z) = \varphi\big(T(z)\big), \quad T(0) = 0. \qquad (3)$$

By proper choices for the degree-weight sequences $(\varphi_\ell)_{\ell\geq 0}$ we obtain the families of $(k+1)$-ary increasing trees and plane-oriented recursive trees.

Example 1. The family of $(k+1)$-ary increasing trees, with integer $k \in \mathbb{N}$, is the family of increasing trees where each node has $k+1$ (labeled) positions for children, going from left to right. Thus, only outdegrees $0, \dots, k+1$ are allowed; moreover, for a node with ℓ children in given order, there are $\binom{k+1}{\ell}$ ways to attach them, see Figure 2. The vacant positions of a node are usually denoted by external nodes, see Figure 1 for an illustration of ternary increasing trees. Hence, the degree-weight generating function of $(k+1)$-ary increasing trees is given by $\varphi(t) = (1+t)^{k+1}$, i.e. $\varphi_\ell = \binom{k+1}{\ell}$, $0 \leq \ell \leq k+1$. Consequently, the degree weight generating function $\varphi(t)$ is given by $\varphi(t) = \sum_{\ell\geq 0}\varphi_\ell t^\ell = (1+t)^k$. By solving the corresponding differential equation (3) one obtains the generating function $T(z) = T(z,k)$ and the numbers $T_n = T_n(k)$[2] of $(k+1)$-ary trees of size n

$$T(z) = \frac{1}{(1-kz)^{\frac{1}{k}}} - 1, \qquad T_n = \prod_{\ell=1}^{n}(k(\ell-1)+1) = k^n\frac{\Gamma(n+1/k)}{\Gamma(1/k)}, \quad n \geq 1.$$

Note that $T_n = Q_n$, the number of k-Stirling permutations of size n (1). For $k = 1$ we obtain the family of binary increasing trees and for $k = 2$ the family of ternary increasing trees.

[2] We usually drop the dependence of $T(z)$ and T_n on k for the sake of simplicity

Example 2. The family of plane-oriented recursive trees consists of rooted plane (=ordered) increasing trees such that all node degrees are allowed with all trees having weight 1. Since plane-oriented recursive trees are ordered trees a new vertex may be joined to an existing vertex v in exactly $\deg^+(v)+1$ positions, where $\deg^+(v)$ denotes the outdegree of node v. These $\deg^+(v)+1$ positions are sometimes represented by external nodes, see Figure 1. Consequently, the total number of positions available to vertex $n+1$ being attached to a tree of size n is given by $\sum_{j=1}^{n}(\deg^+(j)+1) = 2n-1$, independent of the actual shape of the tree of size n. There is exactly one tree of size, $T_1 = 1$. More generally, there are $T_n = \prod_{\ell=1}^{n-1}(2\ell-1) = (2n-3)!!$ different plane-oriented recursive trees of size n, for $n \geq 1$. This number may also be obtained via the formal description above. Since all trees have weight one, we have $\varphi_\ell = 1$ for $\ell \geq 0$, and the degree-weight generating function is given by $\varphi(t) = \sum_{\ell \geq 0} \varphi_\ell t^\ell = \frac{1}{1-t}$. Consequently, by solving the differential equation 3, we get

$$T(z) = 1 - \sqrt{1-2z}, \quad \text{and} \quad T_n = \prod_{\ell=1}^{n-1}(2\ell-1) = (2n-3)!!, \ \text{ for } n \geq 1.$$

Note that $T_{n+1} = (2n-1)!!$, which equals the number of (2-) Stirling permutations of size n and the number of ternary trees of size n. This is no coincidence since Janson has shown that plane-oriented recursive trees of size $n+1$ are in bijection with Stirling permutation of size n, see Theorem 2. Moreover, Janson et al. [15] have recently given a bijection between ternary increasing trees ($k=2$) of size n, as defined above, and plane-oriented recursive trees of size $n+1$.

Remark 1. Both, the family of $(k+1)$-ary increasing trees and the family of plane-oriented recursive trees introduced before can be generated according to tree evolution processes; we refer the interested reader to the work of Panholzer and Prodinger [18] for a comprehensive discussion of the processes; see also Figure 1.

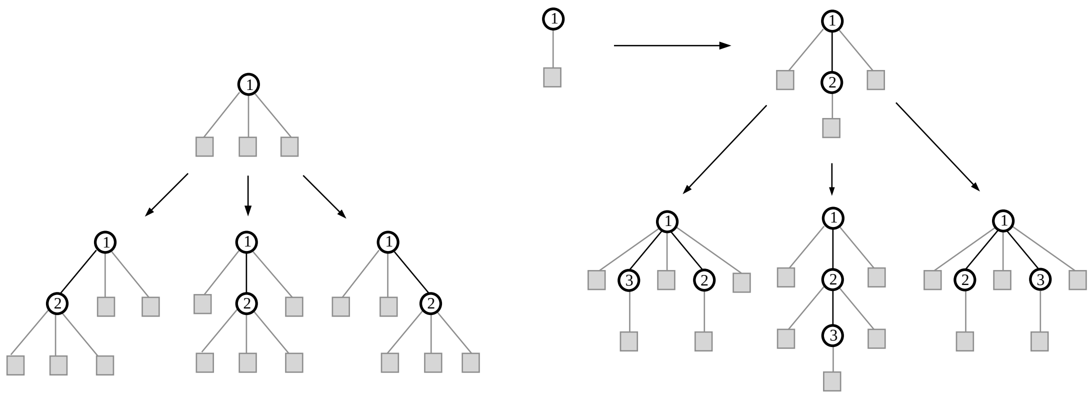

Figure 1: Ternary increasing trees of size one and two and plane-oriented increasing trees of size one, two and three, respectively. The positions where new nodes can be attached are denoted by external nodes.

2.3 Bijections of Gessel and Janson

Theorem 1 (Gessel [19]; see also [15])**.** *Let $k \in \mathbb{N}$. The family $\mathcal{A}_n = \mathcal{A}_n(k+1)$ of $(k+1)$-ary increasing trees of size n is in a natural bijection with k-Stirling permutations, $\mathcal{A}_n(k+1) \cong \mathcal{Q}_n(k)$.*

As shown in [15], the bijection behind Theorem 1 allows to study parameters in k-Stirling permutations via the corresponding parameters in $(k+1)$-ary increasing trees. The bijection is stated explicitly below.

Bijection 1. The depth-first walk of a rooted (plane) tree starts at the root, goes first to the leftmost child of the root, explores that branch (recursively, using the same rules), returns to the root, and continues with the next child of the root, until there are no more children left. We think of $(k+1)$-ary increasing trees, where the empty places are represented by "external nodes". Hence, at any time, any (interior) node has $k+1$ children, some of which may be external nodes. Between these $k+1$ edges going out from a node labeled v, we place k integers v. (External nodes have no children and no labels.) Now we perform the depth-first walk and code the $(k+1)$-ary increasing tree by the sequence of the labels visited as we go around the tree (one may think of actually going around the tree like drawing the contour). In other words, we add label v to the code the k first times we return to node v, but not the first time we arrive there or the last time we return. A $(k+1)$-ary increasing tree of size 1 is encoded by 1^k. A $(k+1)$-ary increasing tree of size n is encoded by a string of $k \cdot n$ integers, where each of the labels $1, \dots, n$ appears exactly k times. In other words, the code is a permutation of the multiset $\{1^k, 2^k, \dots, n^k\}$. Note that for each i, $1 \le i \le n$, the elements occurring between the two occurrences of i are larger than i, since we can only visit nodes with higher labels. Hence the code is a k-Stirling permutation. Moreover, adding a new node $n+1$ at one of the $kn+1$ free positions (i.e., the positions occupied by external nodes) corresponds to inserting the k-tuple $(n+1)^k$ in the code at one of $kn+1$ gaps; note (e.g., by induction) that there is a bijection between external nodes in the tree and gaps in the code. This shows that the code determines the $(k+1)$-ary increasing tree uniquely and that the coding is a bijection.

The inverse, starting with a k-Stirling permutation σ of size n and constructing the corresponding $(k+1)$-ary increasing tree can be described as follows. We proceed recursively starting at step one by decomposing the permutation as $\sigma = \sigma_1 1 \sigma_2 1 \dots \sigma_k 1 \sigma_{k+1}$, where (after a proper relabelling) the σ_i's are again k-Stirling permutations. Now the smallest label in each σ_i is attached to the root node labeled 1. We recursively apply this procedure to each σ_i to obtain the tree representation.

A similar bijection of Janson relates 2-Stirling permutation with plane-oriented recursive trees.

Theorem 2 (Janson [14]). *The family of plane-oriented increasing trees of size $n+1$ is in a natural bijection with 2-Stirling permutations of size n.*

Theorems 1, 2 imply that there exists a bijection between plane-oriented increasing trees of size $n+1$ and ternary increasing tree of size n. As mentioned earlier, a bijection between these tree families was given in [15].

3 Local types in generalized Stirling permutations

It is well known, see Flajolet [7], or Conrad and Flajolet [5], that any ordinary permutation $\tau = \tau_1 \dots \tau_n$ of size n can be classified according to four local order types called peaks (maxima), valleys (minima), double rises and double falls. The classification depends on the relative order of τ_j, with $1 \le j \le n$, to its neighbours, with respect to the border conditions $\tau_0 = -\infty$, $\tau_{n+1} = -\infty$; note that sometimes the border condition $\tau_{n+1} = +\infty$ is used [5], however the condition $\tau_{n+1} = -\infty$ is more consistent with respect to the relation to binary increasing trees. Moreover, due to the bijection with binary increasing trees, there exists a correspondence to certain node types. Below we recall the classification of local types in permutations and node types in binary increasing trees [7], specified according to the index j, with $1 \le j \le n$.

Local type	Peak	Valley	Double rise	Double Fall
Condition	$\tau_{j-1} < \tau_j > \tau_{j+1}$	$\tau_{j-1} > \tau_j < \tau_{j+1}$	$\tau_{j-1} < \tau_j < \tau_{j+1}$	$\tau_{j-1} > \tau_j > \tau_{j+1}$
Node type	Leaf	Double node	Right-branching node	Left-branching node.

A natural question is to extend the notion of local order types to k-Stirling permutations. It turns out that a *variation* of the previous definition of local order types in permutations naturally extends to the general case of k-Stirling permutations. We introduce a slighty different notion of local order types in permutations in the following way.

Definition 1. Given an ordinary permutation $\tau = \tau_1 \dots \tau_n \in \mathcal{S}_n$ and entry i, with $1 \le i \le n$, let j_i denote the index such that $\tau_{j_i} = i$, with $1 \le j_i \le n$. The local order type $L_i(\tau) = \ell_{i,1}\ell_{i,2}$ of entry i in τ, with $1 \le i \le n$, is a string of length 2, with $\ell_{i,1}, \ell_{i,2} \in \{0,1\}$, defined by the relative order of τ_{j_i} to its neighbours $\tau_{j_i-1}, \tau_{j_i+1}$, assuming to the border conditions $\tau_0 = \tau_{n+1} = -\infty$, in the following way. The local type $L_i(\tau)$ of entry i is given by $L_i(\tau) = 00$ if τ_{j_i} is a peak $\tau_{j_i-1} < \tau_{j_i} > \tau_{j_i+1}$, $L_i(\tau) = 11$ a if τ_{j_i} is a valley $\tau_{j-1} > \tau_{j_i} < \tau_{j+1}$, $L_i(\tau) = 01$ if τ_{j_i} is a double rise $\tau_{j_i-1} < \tau_{j_i} < \tau_{j_i+1}$, and $L_i(\tau) = 10$ if τ_{j_i} is double fall $\tau_{j_i-1} > \tau_{j_i} > \tau_{j_i+1}$.

Example 3. The ordinary permutation $\tau = 2534716$ of size seven has the following local types $L_1(\tau) = 11$, $L_2(\tau) = 01$, $L_3(\tau) = 11$, $L_4(\tau) = 01$, $L_5(\tau) = 00$, $L_6(\tau) = 00$, and $L_7(\tau) = 00$.

This new definition readily extends to the general case of k-Stirling permutation, with $k \ge 1$.

Definition 2. Given a k-Stirling permutation $\sigma = \sigma_1\sigma_2 \dots \sigma_{kn}$ of size n, with border conditions $\sigma_0 = \sigma_{nk+1} = -\infty$, let $1 \le j_{i,1} < \dots < j_{i,k} \le kn$ be the indices such that $\sigma_{j_{i,h}} = i$. The local type $L_i(\tau) = \ell_{i,1} \dots \ell_{i,k+1}$ of the numbers i with $1 \le i \le n$ is a string of length $k+1$, with $\ell_{i,1}, \dots, \ell_{i,k+1} \in \{0,1\}$, generated according to relative orders of the $\sigma_{j_{i,h}}$, with $1 \le h \le k+1$, to their neighbors by the following rules.

$$\ell_{i,1} = \begin{cases} 0 & \text{if } \sigma_{j_{i,1}-1} < \sigma_{j_{i,1}}, \\ 1 & \text{if } \sigma_{j_{i,1}-1} > \sigma_{j_{i,1}}; \end{cases} \qquad \ell_{i,k+1} = \begin{cases} 0 & \text{if } \sigma_{j_{i,k}} > \sigma_{j_{i,k}+1}, \\ 1 & \text{if } \sigma_{j_{i,k}} < \sigma_{j_{i,k}+1}; \end{cases}$$

and

$$\ell_{i,h} = \begin{cases} 0 & \text{if } \sigma_{j_{i,h}-1} = \sigma_{j_{i,h}}, \\ 1 & \text{if } \sigma_{j_{i,h}-1} \neq \sigma_{j_{i,h}}; \end{cases} \qquad \text{for } h \le 2 \le k.$$

Example 4. The 3-Stirling permutation $\sigma = 112233321445554666$ of size six has the following local types $L_1(\sigma) = 0011$, $L_2(\sigma) = 0010$, $L_3(\sigma) = 0000$, $L_4(\sigma) = 0010$, $L_5(\sigma) = 0000$, $L_6(\sigma) = 0000$.

Since there are exactly 2^{k+1} different possible local types, we obtain the following result.

Proposition 1. *A k-Stirling permutation $\sigma = \sigma_1\sigma_2 \dots \sigma_{kn}$ of size n of the multiset $\{1^k, 2^k, \dots, n^k\}$ can be classified according to 2^{k+1} different local types, with respect to the local rules in Definition 2.*

Next we want to relate the local types in k-Stirling permutations to node types in $(k+1)$-ary increasing trees. By definition of $(k+1)$-ary increasing trees, every node has exactly $k+1$ (labeled) positions for children. Some of the $k+1$ positions may be occupied by (internal) nodes, some other may be vacant (occupied by external nodes). We propose the following definition.

Definition 3. The node labeled i, with $1 \le i \le n$, in a $(k+1)$-ary increasing trees T of size n may be specified according to the structure of its children, i.e. a sequence $G_i(T) = g_{i,1} \dots, g_{i,k+1} \in \{0,1\}^{k+1}$ of length $k+1$, where $g_{i,h} \in \{0,1\}$ specifies whether the h-th position from node i, going from left to right, is occupied by a node, $g_{i,h} = 1$, or not, $g_{i,h} = 0$, in the $(k+1)$-ary increasing tree of size n, $1 \le h \le k+1$.

In other words, $g_{i,h}$ encodes whether node i has an internal children via its h-th edge, going from left to right, or not.

Example 5. In the case $k = 1$, binary increasing trees, we have $4 = 2^2$ different types of nodes. We already observed that sequence 11 corresponds to a double node, the sequence 10 to a left-branching node, the sequence 01 to a right-branching node, and the sequence 00 to a leaf.

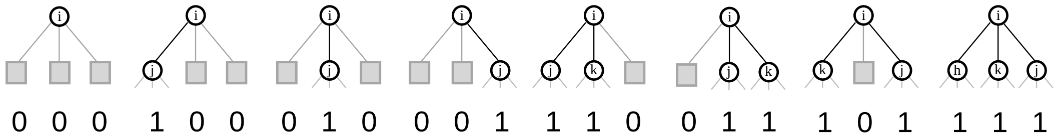

Figure 2: The eight different node types in ternary increasing trees, assuming that $j, h, k > i \geq 1$.

Example 6. In the case $k = 2$, ternary increasing trees, we have $8 = 2^3$ different types of nodes. The sequence 111 corresponds to a triple node, 101 to a (left,right)-branching node, 110 to a (left,center)-branching node, 011 to a (center,right)-branching node, 100 to a left-branching node, 010 to a center-branching node, 001 to a right-branching node, and 000 to a leaf, respectively. See Figure 2 for an illustration.

Theorem 3. *By Bijection 1 the local types $L_i(\sigma)$ in a k-Stirling permutation $\sigma = \sigma_1 \dots \sigma_{kn}$ of size n coincide with the node types $G_i(T)$ of the corresponding $(k+1)$-ary increasing trees T of size n, $L_i(\sigma) = G_i(T)$, $1 \leq i \leq n$.*

Proof. We use Theorem 1 and the bijection between k-Stirling permutations and $(k+1)$-ary increasing trees, which is based on a depth-first walk as described in Bijection 1. We start the depth first walk at the root of a given $(k+1)$-ary increasing tree T of size n with node types $G_1(T), \dots, G_n(T)$, and construct the corresponding k-Stirling permutation $\sigma = \sigma(T)$ of size n by traversing the tree T according to Bijection 1. We show that the local order type $L_i(\sigma) = \ell_{i,1} \dots \ell_{i,k+1}$ equals the node type $G_i(T) = g_{i,1} \dots g_{i,k+1}$ of the node labeled i, for all $1 \leq i \leq n$. Assume first that the first of the $k+1$ positions of the node labeled i is vacant, the node degree type $g_{i,1} = 0$. By Bijection 1 we observe that $g_{i,1} = 0$ implies that $i = \sigma_{j_{i,1}} > \sigma_{j_{i,1}-1} = m$ and consequently $\ell_{i,1} = \ell_{i,1}(\sigma) = 0$; here $j_{i,1}$ denotes the index of the first occurrence of i in the corresponding k-Stirling permutation $\sigma = \sigma_1 \dots \sigma_{kn}$, since a smaller number $m < i$ must have been observed earlier according to the depth-first walk and the property that the tree is increasingly labeled.

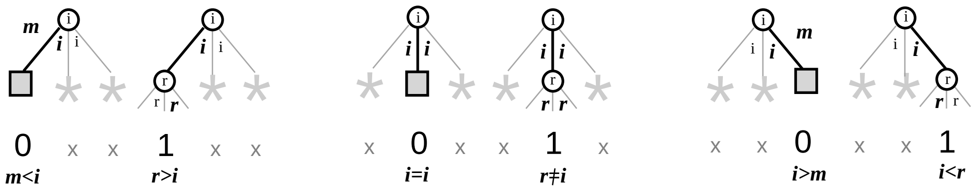

Figure 3: A schematic representation of the correspondence between local order types in 2-Stirling permutations and local node types in ternary increasing trees according to Bijection 1: the tree is traversed according to a depth-first walk, assuming that $r > i$ and $m < i$.

Assume now that the converse is true $g_{i,1} = 1$. By the depth-first walk and the definition of increasing trees we have $i = \sigma_{j_{i,1}} < \sigma_{j_{i,1}-1}$. More generally let $1 \leq j_{i,1} < \dots < j_{i,k} \leq kn$ denote the k indices of the occurrences of i in the corresponding k-Stirling permutation σ, $\sigma_{j_{i,h}} = i$ for $1 \leq h \leq k$. For $2 \leq h \leq k$ we note that $g_{i,h} = 0$ implies that the indices $j_{i,h}$ and $j_{i,h+1}$ satisfy $j_{i,h} + 1 = j_{i,h+1}$ and consequently $\sigma_{j_{i,h}} = \sigma_{j_{i,h}+1} = i$ and further $\ell_{i,h} = 0$; the converse is also true. Finally, if $g_{i,k+1} = 0$ the $\sigma_{j_{i,k+1}} > \sigma_{j_{i,k+1}+1}$ and consequently $\ell_{i,k+1} = 0$. The converse is easily be seen to be true. $\square$

3.1 Local types and path diagrams

It is well known by a theorem of Françon and Viennot [9] (see also Flajolet [7]) that the description of ordinary permutations via local types is closely related to path diagrams. In the following we give a bijection between k-Stirling permutations ($(k+1)$-ary increasing trees) and path diagrams with $k+2$ different step-vectors. First we recall some definitions of [7] concerning lattice paths. We have $k+2$ step vectors, consisting of k different rise vectors $a_1, \dots, a_k$, with $a_\ell = (1, \ell)$ for $1 \le \ell \le k$, a fall vector $b = (1, -1)$, and a level vector $c = (1, 0)$. To each word $u = u_1 \dots u_n$ on the alphabet $\mathcal{A} = \{a_1, \dots, a_k, b, c\}$ there exists an associated sequence of points $M_0 M_1 \dots M_n$, with $M_0 = (0, 0)$ such that $M_j = M_{j-1} + u_j$, for all $1 \le j \le n$. We only consider paths that are positive and ending at $(0, 0)$, corresponding to sequences where that all the points have a non-negative y-coordinate, and $M_n = M_0 = (0, 0)$.

In a labeled path each step is indexed with the height of the point from which it starts. For a positive path associated to the word $u = u_1 \dots u_n$ with corresponding sequence of points $M_0 M_1 \dots M_n$ with $M_i = (x_i, y_i)$, the labeling $\lambda(u)$ is a word of length n over the infinite alphabet X,

$$X = \{b_0, b_1, b_2 \dots\} \cup \{c_0, c_1, c_2 \dots\} \cup \bigcup_{\ell=1}^{k} \{a_{0,\ell}, a_{1,\ell}, a_{2,\ell} \dots\},$$

by $\lambda(u) = v_1 \dots v_n$ via the following rules

$$\begin{aligned} (i) \quad & \text{if } u_j = a_\ell, \quad \text{then } v_j = a_{y_{j-1},\ell}, \\ (ii) \quad & \text{if } u_j = b, \quad \text{then } v_j = b_{y_{j-1}}, \\ (iii) \quad & \text{if } u_j = c, \quad \text{then } v_j = c_{y_{j-1}}. \end{aligned}$$

Next we recall the definition of path diagrams (see i.e. [9], [7] and the references therein). A system of path diagrams on a given set of (labeled) paths is defined as follows. A path diagram is a couple $(\lambda(u), s)$, where $u = u_1 \dots u_n$ is a path, and s is a sequence of integers $s = s_1 \dots s_n$ such that for all $j : 0 \le s < \text{pos}(v_j)$, where pos $=: X \to \mathbb{N}$ is called a possibility function.

Now we are ready to state the connection between path diagrams, k-Stirling permutations and $(k+1)$-ary increasing trees.

Theorem 4. *The class of k-Stirling permutations of size $n+1$ (the family of $(k+1)$-ary increasing trees of size $n+1$) is in bijection with path diagrams of length n, with possibility function* pos(.) *given by*

$$\text{pos}(a_{j,\ell}) = \binom{k+1}{\ell+1}(j+1), \quad 1 \le \ell \le k, \quad \text{pos}(b_j) = j+1, \quad \text{pos}(c_j) = (k+1)(j+1),$$

with respect to the labeled paths induced by the family of $k+2$ step vectors $a_1, \dots, a_k, b, c$, with rise vectors $a_\ell = (1, \ell)$ for $1 \le \ell \le k$, fall vector $b = (1, -1)$, and level vector $c = (1, 0)$.

Remark 2. For $k = 1$ this reduces to the classical correspondence of Françon and Viennot [9]. Note that one may interpret c as a rise vector, corresponding to the case $\ell = 0$, which would simplify the presentation. However, due to the importance of the case $k = 1$ we opted not to do so, in order to be coherent with the presentations of [9], [7].

Proof. Following [7] we shall set $a_\ell = \sum_{i=1}^{\binom{k+1}{\ell+1}} a_\ell^{(i)}$, with $\text{pos}(a_{j,\ell}^{(i)}) = j+1$, for $1 \le i \le \binom{k+1}{\ell+1}$ and $1 \le \ell \le k$; moreover we also set $c = \sum_{i=1}^{k+1} c^{(i)}$, with $\text{pos}(c_j^{(i)}) = j+1$ for $1 \le i \le k+1$. We readily observe that the new refined path diagrams are in bijection with the previously defined path diagrams, with possibility function given by

$$\text{pos}(a_{j,\ell}) = \binom{k+1}{\ell+1}(j+1), \quad 1 \le \ell \le k, \quad \text{pos}(b_j) = j+1, \quad \text{pos}(c_j) = (k+1)(j+1).$$

We recursively construct a $(k+1)$-ary increasing tree, starting from a path diagram $(\lambda(u), s)$, with $\lambda(u) = v_1 \dots v_n$ as follows. At step zero we start with the empty tree and one position to insert a node. At step j, $1 \le j \le n$, we insert node j to one of the vacant positions, where the number h_j of vacant positions at step j is given by the height of the path at position j plus one. If letter $v_j = a^{(i)}_{h_{j-1},\ell}$, with $1 \le i \le \binom{k+1}{\ell+1}$ and $1 \le \ell \le k$, then node j is assumed to have outdegree $\ell+1$. The specific outdegree structure of node j, the distribution of the $\ell+1$ children to $k+1$ possible places, is, according to Definition 3, determined by an arbitrary but fixed bijection from the set $\{h = h_1 \dots h_{k+1} \in \{0,1\}^{k+1} \mid \sum_{i=1}^{k+1} h_i = \ell+1\}$ to the set $\{a^{(i)}_\ell \mid 1 \le i \le \binom{k+1}{\ell+1}\}$. If the number in the possibility sequence is s_j, we assign node j at the $1+s_j$ vacant position starting from the left. The construction is terminated by putting node $(n+1)$ as a leaf in the last vacant position after stage n. $\square$

Example 7. Consider the case $k=2$, corresponding to ternary increasing trees, or equivalently Stirling permutations. Below we illustrate the procedure stated above on the path diagram $v = a_{0,2}a^{(2)}_{2,1}b_3b_2c^{(3)}_1b_1$ and $s = 0030011$, assuming the local outdegree structure correspondence determined by $a^{(1)}_1 \sim 110$, $a^{(2)}_1 \sim 101$, $a^{(3)}_1 \sim 011$, $c^{(1)} \sim 100$, $c^{(2)} \sim 010$, $c^{(3)} \sim 001$. By using for $k=2$ the

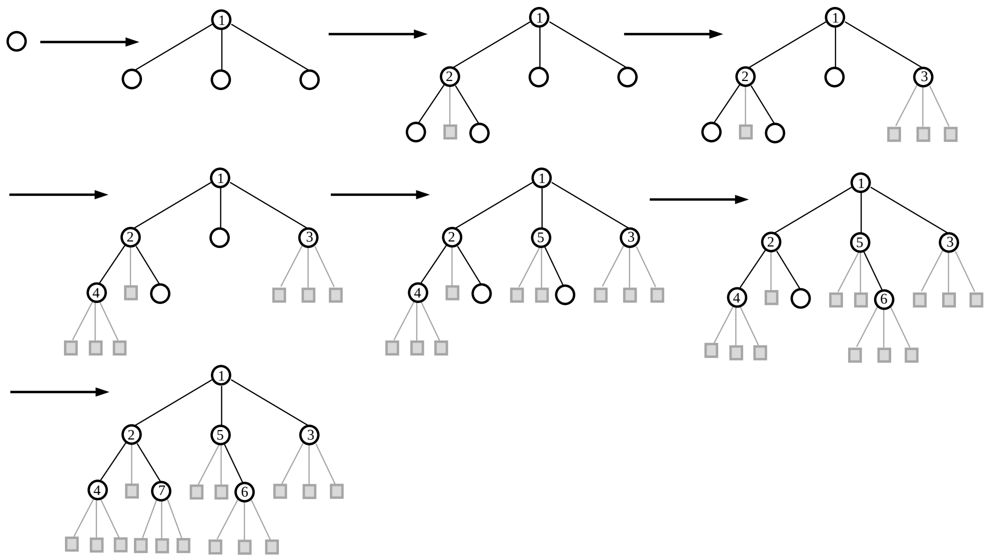

Figure 4: An illustration of the recursive construction of a ternary increasing trees, with respect to the path diagram $(\lambda(u), s)$ with $\lambda(u) = v = a_{0,2}a^{(2)}_{2,1}b_3b_2c^{(3)}_1b_1$ and $s = 0030011$

bijection between $(k+1)$-ary increasing trees and k-Stirling permutations, see [15], we immediately obtain the corresponding Stirling permutation σ of size seven, $\sigma = 44227715566133$. Note that we can also directly construct the Stirling permutation, since the local types of the outdegree of the nodes in the ternary increasing tree correspond to the local types of the numbers in the permutation. One may think of this procedure as some kind of "flattening of the tree to a line", see below and compare with the sequence of trees in Figure 4.

$$\circ \to \circ 1 \circ 1 \circ \to \circ 22 \circ 1 \circ 1 \circ \to \circ 22 \circ 1 \circ 133 \to 4422 \circ 1 \circ 133 \to 4422 \circ 155 \circ 133$$
$$\to 442277155 \circ 133 \to 44227715566133.$$

3.2 A continued fraction type representation of local types

Flajolet [7] used the correspondence between path diagrams and formal power series to obtain continued fraction representations of the generating functions of many parameters in ordinary permutations. More precisely, in the context of permutations and binary increasing trees he derived, amongst many other results, a continued fraction representation of generating function of local types in permutations, or equivalently of node types in binary increasing trees. We will use the methods of [7] and the beforehand proven path diagram representation of k-Stirling permutations and $(k+1)$-ary increasing trees to obtain a continued fraction type representation of the generating function of the local types, and consequently also of node types. First we have to recall some more definitions of the work [7] concerning formal power series. Let $C\langle\!\langle X\rangle\!\rangle$ denote the monoid algebra of formal power series $s=\sum_{u\in X^*} s_u\cdot u$ on the set of non-commutative variables (alphabet) X with coefficients in the field of complex numbers, with sums and Cauchy products are defined in the usual way

$$s+t=\sum_{u\in X^*}(s_u+t_u)\cdot u, \qquad s\cdot t=\sum_{u\in X^*}\Bigl(\sum_{vw=u} s_v t_w\Bigr)\cdot u.$$

In order to define the convergence of a series, one introduces the valuation of a series $\operatorname{val}(s)$, defined by

$$\operatorname{val}(s)=\min\{|u| : s_u\neq 0\},$$

where $|u|$ denotes the length of the word $u\in X^*$. A sequence of elements $(s_n)_{n\in\mathbb{N}}$, $s_n\in C\langle\!\langle X\rangle\!\rangle$, converges to a limit $s\in C\langle\!\langle X\rangle\!\rangle$ if

$$\lim_{n\to\infty}\operatorname{val}(s-s_n)=\infty.$$

Multiplicative inverses exist for series having a constant term different from zero; for example $(1-u)^{-1}=\sum_{\ell\geq 0}u^\ell$, where $(1-u)^{-1}$ is known as the quasiinverse of u. Note that we will subsequently use the notation $(u|v)/w=uw^{-1}v$. The characteristic series $\operatorname{char}(S)$ of $S\subset X^*$ is defined as

$$\operatorname{char}(S)=\sum_{u\in S}u.$$

Finally, following [7] we use for subsets E,F of X^* the alternative notations $E+F$ for the union $E\cup F$, $E\cdot F$ for the extension to sets of the catenation operation on words, and let $E^*=\epsilon+E+E\cdot E+E\cdot E\cdot E+\dots$, with ϵ denoting the empty word. Moreover, we will use a Lemma (Lemma 1 of Flajolet [7]), which allows to translate operations on sets of words into corresponding operations on series, provided certain non ambiguity conditions are satisfied.

Lemma 1. *Let E, F be subsets of X^*. Then*

1. $\operatorname{char}(E+F)=\operatorname{char}(E)+\operatorname{char}(F)$ *provided* $E\cap F=\emptyset$,
2. $\operatorname{char}(E\cdot F)=\operatorname{char}(E)\cdot\operatorname{char}(F)$ *provided that* $E\cdot F$ *has the unique factorization property,* $\forall u,u'\in E\ \forall v,v'\in F\ uv=u'v'$ *implies* $u=u'$ *and* $v=v'$,
3. $\operatorname{char}(E^*)=(1-\operatorname{char}(E))^{-1}$ *provided the following two condition hold:* $E^j\cap E^k=\emptyset\ \forall j,k$ *with* $j\neq k$, *each* E^k *has the unique factorization property.*

With the help of Lemma 1 one can translate operations on sets of words into corresponding operations on series provided certain non-ambiguity conditions are satisfied.

Let $C_i^{[h]}=C_i^{[h]}(k)$ be defined as the characteristic series of all labeled paths with step vectors given by $a_1,\dots,a_k,b,c$ starting and ending at the level i, with $i\geq 0$, never going below level i and above level $i+h$, with $h\geq 0$. We assume that formal convention $C_i^{[h]}=0$ if $h<0$. Moreover, let $C^{[h]}=C_0^{[h]}$.

We introduce the notation $\left\langle C_i^{[h]} \right\rangle_1 := (a_{i,1}|b_{i+1})C_{i+1}^{[h-1]}$, $\left\langle C_i^{[h]} \right\rangle_2 := ((a_{i,2}|b_{i+2}) \cdot C_{i+2}^{[h-2]}|b_{i+1}) \cdot C_{i+1}^{[h-1]}$, and in general for integer $1 \leq \ell \leq k$ let $\left\langle C_i^{[h]} \right\rangle_\ell$ be defined by

$$\left\langle C_i^{[h]} \right\rangle_\ell = (\dots((a_{i,\ell}|b_{i+\ell})C_{i+\ell}^{[h-\ell]}|b_{i+\ell-1})C_{i+\ell-1}^{[h-(\ell-1)]} \dots |b_{i+1})C_{i+1}^{[h-1]}.$$

Proposition 2. *The characteristic series $C_i^{[h]} = C_i^{[h]}(k)$ of all labeled paths with step vectors given by $a_1, \dots, a_k, b, c$ starting and ending at the level i, with $i \geq 0$, never going below level i and above level $i + h$, with $h \geq 0$, satisfies*

$$C_i^{[h]} = \frac{1}{1 - c_i - \sum_{\ell=1}^{k} \left\langle C_i^{[h]} \right\rangle_\ell}.$$

The double sequence $(C_i^{[h]})_{i,h\geq 0}$ converges for $h \to \infty$. Its limit $(C_i)_{i\geq 0}$ given as follows.

$$C_i = \frac{1}{1 - c_i - \sum_{\ell=1}^{k} \left\langle C_i \right\rangle_\ell}.$$

In particular, $C = C_0$ equals the characteristic sequence of all labeled paths $\mathcal{P}$, starting and ending at the x-axis, never going below the y-axis, with step vectors given by $a_1, \dots, a_k, b, c$.

Remark 3. The case $k = 1$, treated by Flajolet [7], corresponds to binary increasing trees and ordinary permutation.

Proof. For the sake of simplicity we only present the proof of the special case $k = 2$, corresponding to Stirling permutations and ternary increasing trees. We prove that

$$C_0^{[h]} = \frac{1}{1 - c_0 - \sum_{\ell=1}^{2} \left\langle C_0^{[h]} \right\rangle_\ell} = \frac{1}{1 - c_0 - (a_{0,1}|b_1)C_1^{[h-1]} - ((a_{0,2}|b_2)C_2^{[h-2]}|b_1)C_1^{[h-1]}}$$

equals the characteristic series of the set $\mathcal{P}^{[h]}$ of all labeled paths with step vectors a_1, a_2, b, c, starting and ending at level zero with height bounded by h. More generally, for $i \geq 0$

$$C_i^{[h]} = \frac{1}{1 - c_i - \sum_{\ell=1}^{2} \left\langle C_i^{[h]} \right\rangle_\ell} = \frac{1}{1 - c_i - (a_{i,1}|b_{i+1})C_{i+1}^{[h-1]} - ((a_{i,2}|b_{i+2})C_{i+2}^{[h-2]}|b_{i+1})C_{i+1}^{[h-1]}}$$

equals the characteristic series of all labeled paths starting and ending at level i with height bounded by h. Note that by our previous notation $(u|v)/w = uw^{-1}v$ and $(1 - u)^{-1} = \sum_{\ell\geq 0} u^\ell$ regarding quasiinverse series, we have for instance

$$\begin{aligned}(a_{i,1}|b_{i+1})C_{i+1}^{[h-1]} &= a_{i,1}\big(C_{i+1}^{[h-1]}\big)^{-1} b_{i+1} \\ &= a_{i,1} \sum_{\ell\geq 0} \Big(c_{i+1} + (a_{i+1,1}|b_{i+2})C_{i+2}^{[h-2]} + ((a_{i+1,2}|b_{i+3})C_{i+3}^{[h-3]}|b_{i+2})C_{i+2}^{[h-2]} b_{i+2}\Big)^\ell b_{i+1}.\end{aligned}$$

For the first few values of $h = 1, 2, 3$ we obtain

$$\begin{aligned}
\mathcal{P}^{[0]} &= (c_0)^* \\
\mathcal{P}^{[1]} &= (c_0 + a_{0,1}c_1^* b_1)^* \\
\mathcal{P}^{[2]} &= (c_0 + a_{0,1}(c_1 + a_{1,1}c_2^* b_2)^* b_1 + a_{0,2}c_2^* b_2(c_1 + a_{1,1}c_2^* b_2)^* b_1)^* \\
\mathcal{P}^{[3]} &= \Big(c_0 + a_{0,1}\big(c_1 + a_{1,1}(c_2 + a_{2,1}c_3^* b_3)^* b_2 + a_{1,2}c_3^* b_3(c_2 + a_{2,1}c_3^* b_3)^* b_2\big)^* b_1 \\
&\quad + a_{0,2}(c_2 + a_{2,1}c_3^* b_3)^* b_2\big(c_1 + a_{1,1}(c_2 + a_{2,1}c_3^* b_3)^* b_2 + a_{1,2}c_3^* b_3(c_2 + a_{2,1}c_3^* b_3)^* b_2\big)^* b_1\Big)^*.
\end{aligned}$$

In order to simplify the recursive description of $\mathcal{P}^{[h]}$ we introduce the refined sets $\mathcal{P}_i^{[h]}$ consisting of the paths starting and ending at level i with height bounded by h, where $\mathcal{P}_0^{[h]} = \mathcal{P}^{[h]}$. Note that $\mathcal{P}_i^{[h]}$ can easily be obtained from $\mathcal{P}_0^{[h]} = \mathcal{P}^{[h]}$ by shifting the first index encoding the level by i; we have $\mathcal{P}_i^{[0]} = (c_i)^*$, $\mathcal{P}_i^{[1]} = (c_i + a_{i,1}c_{i+1}^* b_i)^*$. We can write $\mathcal{P}^{[h]}$ in the following way.

$$\begin{aligned}
\mathcal{P}^{[0]} &= (c_0)^* \\
\mathcal{P}^{[1]} &= (c_0 + a_{0,1}\mathcal{P}_{[0],1}b_1)^* \\
\mathcal{P}^{[2]} &= (c_0 + a_{0,1}\mathcal{P}_{[1],1}b_1 + a_{0,2}\mathcal{P}_{[0],2}b_2\mathcal{P}_{[1],1}b_1)^* \\
\mathcal{P}^{[3]} &= \Big(c_0 + a_{0,1}\mathcal{P}_{[2],1}b_1 + a_{0,2}\mathcal{P}_{[1],2}b_2\mathcal{P}_{[2],1}b_1\Big)^*.
\end{aligned}$$

By induction one can prove that the following unambiguous description of $\mathcal{P}^{[h]}$.

$$\mathcal{P}^{[h]} = \Big(c_0 + a_{0,1}\mathcal{P}_{[h-1],1}b_1 + a_{0,2}\mathcal{P}_{[h-2],2}b_2\mathcal{P}_{[h-1],1}b_1\Big)^*.$$

More generally, we have

$$\mathcal{P}_i^{[h]} = \Big(c_i + a_{i,1}\mathcal{P}_{i+1}^{[h-1]}b_{i+1} + a_{i,2}\mathcal{P}_{i+2}^{[h-2]}b_{i+2}\mathcal{P}_{i+1}^{[h-1]}b_{i+1}\Big)^*.$$

Since $\mathcal{P}_i^{[h]}$ is obtained from $\mathcal{P}_0^{[h]} = \mathcal{P}^{[h]}$ by an index shift, we recursively obtain the stated description of $C_0^{[h]}$ by replacing the operations $+, \cdot, *$ on the sets of words by the series operations $+, |$, and quasi-inverse. Moreover, the characteristic series of the refined sets $\mathcal{P}_i^{[h]}$ is simply given by $C_i^{[h]}$. One observes the inclusion

$$\mathcal{P}^{[0]} \subset \mathcal{P}^{[1]} \subset \mathcal{P}^{[2]} \subset \dots \subset \mathcal{P},$$

or more generally

$$\mathcal{P}_i^{[0]} \subset \mathcal{P}_i^{[1]} \subset \mathcal{P}_i^{[2]} \subset \dots \subset \mathcal{P}_i \quad i \geq 0.$$

Since paths of height h have at least length greater or equal $\lceil \frac{h}{k} \rceil$, with $k = 2$ in the presented case corresponding to ternary increasing trees and Stirling permutations, we have

$$\mathrm{val}(C_i - C_i^{[h-1]}) \geq \lceil \frac{h}{k} \rceil,$$

and consequently

$$\lim_{h \to \infty} C_i^{[h]} = C_i.$$

The proof of the general case $k > 2$ is similar but more involved. □

Subsequently, we will enumerate k-Stirling permutations according to the 2^{k+1} different local types. Let $P_{\mathbf{m}_0,\dots,\mathbf{m}_{k+1}}$ denote the number of k-Stirling permutations $\sigma \in \mathcal{Q}$, where the 2^{k+1} local types are specified according to $\mathbf{m}_i = (m_{i,1}, \dots, m_{i,\binom{k+1}{i}})$, $0 \leq i \leq k+1$. The generating function $P(\mathbf{z}_0, \dots, \mathbf{z}_{k+1}, t)$ of k-Stirling permutations with respect to the 2^{k+1} local types, or equivalently $(k+1)$-ary increasing trees with respect to the 2^{k+1} different node types, is defined by

$$P(\mathbf{z}_0, \dots, \mathbf{z}_{k+1}, t) = \sum_{\mathbf{m}_0,\dots,\mathbf{m}_{k+1}} P_{\mathbf{m}_0,\dots,\mathbf{m}_{k+1}} \mathbf{z_0}^{\mathbf{m}_0} \dots \mathbf{z_{k+1}}^{\mathbf{m}_{k+1}} t^{\sum_{i=0}^{k+1}\sum_{\ell=1}^{\binom{k+1}{i}} m_{i,\ell}}..$$

Now we can state the main result of this section, namely the continued fraction representation of the generating function of local types in k-Stirling permutations and node types in $(k+1)$-ary increasing trees.

Theorem 5. *The generating function $P(\mathbf{z}_0,\dots,\mathbf{z}_{k+1})$ of k-Stirling permutations, or equivalently $(k+1)$-ary increasing trees, with respect to the 2^{k+1} local types,*

$$P(\mathbf{z}_0,\dots,\mathbf{z}_{k+1},t)=\sum_{\mathbf{m}_0,\dots,\mathbf{m}_{k+1}} P_{\mathbf{m}_0,\dots,\mathbf{m}_{k+1}}\mathbf{z_0}^{\mathbf{m}_0}\dots\mathbf{z_{k+1}}^{\mathbf{m}_{k+1}}t^{\sum_{i=0}^{k+1}\sum_{\ell=1}^{\binom{k+1}{i}}m_{i,\ell}}$$

is given by

$$P(\mathbf{z}_0,\dots,\mathbf{z}_{k+1},t)=\cfrac{1}{1-1\sum_{\ell=1}^{k+1}tz_{1,\ell}-\cfrac{1\cdot 2\sum_{\ell=1}^{\binom{k+1}{2}}z_{2,\ell}t^2z_{0,\ell}}{1-2\sum_{\ell=1}^{k+1}tz_{1,\ell}-\cfrac{2\cdot 3\sum_{\ell=1}^{\binom{k+1}{2}}t^2z_{2,\ell}z_{0,\ell}}{\dots}-\dots}-\dots}.$$

Corollary 1. *An expansion of the generating function $\sum_{n\ge 0}k^{n+1}\frac{\Gamma(n+1+\frac{1}{k})}{\Gamma(\frac{1}{k})}t^n$ is obtained from the generating function $P(\mathbf{z}_0,\dots,\mathbf{z}_{k+1},t)$ by setting $\mathbf{z}_\ell=(1,\dots,1)$, $0\le\ell\le k+1$. In particular, we obtain for $k=2$ the identity*

$$\sum_{n\ge 0}(2n+1)!!\,t^n=\cfrac{1}{1-1\cdot\binom{3}{1}t-\cfrac{1\cdot2\cdot\binom{3}{2}t^2}{1-2\cdot\binom{3}{1}t-\cfrac{2\cdot3\cdot\binom{3}{2}t^2}{1-3\cdot\binom{3}{1}t\dots}-\cfrac{2\cdot1\cdot2\cdot3\cdot\binom{3}{3}t^3}{1-4\cdot\binom{3}{1}t\dots}}-\cfrac{1\cdot2\cdot3\cdot\binom{3}{3}t^3}{\left(1-3\cdot\binom{3}{1}t\dots\right)\left(1-2\cdot\binom{3}{1}t\dots\right)}}$$
$$=1+3t+15t^2+105t^3+945t^4+\dots$$

Remark 4. Below each fraction bar in the continued fraction type representation of the formal power series there are $k+1$ terms, starting with 1. As mentioned earlier the case $k=1$ is a result of Flajolet [7].

Proof. We combine our earlier results. Theorem 3 shows that we can use the same representation for local types of in k-Stirling permutations and node types in $(k+1)$-ary increasing trees. An application of Theorem 4 and Proposition 2, together with the morphism $\mu: C\langle\langle X\rangle\rangle\to C[[\mathbf{z}]]$ with $\mu(a_{j,\ell}^{(i)})=(j+1)tz_{\ell+1,i}$, $1\le\ell\le k$ and $1\le i\le\binom{k+1}{\ell+1}$, $\mu(c_j^{(i)})=(j+1)tz_{1,i}$, $1\le i\le k+1$ and $\mu(b_j)=(j+1)tz_0$, proves the result stated in Theorem 5. The result of Corollary 1 follows by setting $\mathbf{z}_\ell=(1,\dots,1)$ for $0\le\ell\le k+1$. $\square$

3.3 Path diagrams and plane-oriented increasing trees

In the special case $k=2$ Janson [14] showed that the class of 2-Stirling permutations of size n is in bijection with the class of plane-oriented increasing trees of size $n+1$. A bijection between ternary increasing tree of size n and plane-oriented increasing trees of size $n+1$ was given in [15]. We will provide a bijection between path diagrams with an infinite number of rise vectors, plane-oriented increasing trees and Stirling permutations. For the path diagram description we proceed as in Subsection 3.1: First we introduce a family of step vectors; then we state the bijection between plane-oriented increasing trees and path diagrams with suitably defined possibility function. The family of step vectors consists of an infinite number of rise vectors $\mathbf{a}=(a_\ell)_{\ell\in\mathbb{N}}$, with $a_\ell=(1,\ell)$ for $\ell\in\mathbb{N}$, a fall vector $b=(1,-1)$, and a level vector $c=(1,0)$. To each word $u=u_1\dots u_n$ on the alphabet $\mathcal{A}=\{\mathbf{a},b,c\}$ there exists an associated sequence of points $M_0M_1\dots M_n$, with $M_0=(0,0)$ such that $M_j=M_{j-1}+u_j$, for all $1\le j\le n$. As before, we only consider paths that are positive and ending at $(0,0)$, $M_n=M_0$. Moreover, we label the paths according to their vertical positions. Note that the path diagram representation of plane-oriented increasing trees encodes the tree via their outdegree distribution.

Theorem 6. *The class of plane-oriented increasing tree of size $n+1$ is in bijection with path diagrams of length n, with possibility function* pos *given by*

$$\mathrm{pos}(a_{j,\ell}) = j+1, \quad \ell \in \mathbb{N}, \quad \mathrm{pos}(b_j) = j+1, \quad \mathrm{pos}(c_j) = j+1,$$

with respect to the infinite family of step vectors defined previously.

Proof. We use again a recursive construction to obtain, starting from a path diagram $(\lambda(u), s)$ with $\lambda(u) = v_1 \dots v_n$ of length n, the corresponding plane-oriented increasing tree of size n. At step zero we start with the empty tree and one position to insert a node. At step j, $1 \le j \le n$, we insert node j to one of the vacant positions, where the number h_j of vacant positions at step j is given by the height of the path at position j plus one. If letter $v_j = a_{h_{j-1},\ell}$, with $\ell \in \mathbb{N}$, then node j is assumed to have outdegree $\ell + 1$. In the case of $v_j = b_{h_{j-1}}$ or $v_j = c_{h_{j-1}}$, then node j is assumed to have outdegree zero or one, respectively. If the number in the possibility sequence is s_j, we assign node j at the $1 + s_j$ vacant position starting from the left. The construction is terminated by putting node $(n+1)$ as a leaf in the last vacant position after stage n. $\square$

Let $X_{n,j}$ denote the number of nodes of outdegree j in a random plane-oriented increasing tree of size n. We relate the distribution of outdegrees to suitably defined statistics in ternary increasing tree and Stirling permutations. Any ternary increasing tree can be decomposed by deleting all center edges into trees having only left or right edges. The original tree is reobtained by connecting the arising left-right trees using center edges. Let $X_{n,j}^{[LR]}$ denote the number of size j left-right trees in a random ternary increasing tree of size n.

Concerning Stirling permutations $\sigma = \sigma_1 \dots \sigma_{2n}$ we introduce the parameter sub-block structures of size j as follows. A block in a Stirling permutation σ is a substring $\sigma_p \cdots \sigma_q$ with $\sigma_p = \sigma_q$ that is maximal, i.e. not contained in any larger such substring [15]. There is obviously at most one block for every $i = 1, \dots, n$, extending from the first occurrence of i to the last; we say that i forms a block when this substring really is a block, i.e. when it is not contained in a string $\ell \cdots \ell$ for some $\ell < i$. Assume that σ can be decomposed into ℓ blocks, $\sigma = [B_1][B_2] \dots [B_\ell]$. Remove in each of the blocks the left and rightmost number. We are left with subblocks, possibly empty, which are after an order preserving relabeling again (sub-)Stirling permutations. We recursively determine again the (sub)-block structure in the new Stirling permutations. Let $X_{n,j}^{[B]} = X_{n,j}^{[B]}(\sigma)$ denotes the number of (sub-)blocks equal to j in a size n Stirling permutation σ, considering all (sub)-blocks obtained by the recursive process described before.

Example 8. The Stirling permutation $\sigma = 221553367788614499$ of size nine has block decomposition $\sigma = [22][155336778861][44][99]$. After removal of the left and rightmost entries in the blocks the only non-empty subblock or (sub-)Stirling permutation is given by 5533677886. After an order preserving relabeling we get $\sigma' = 2211344553$. We have $\sigma' = [22][11][344553]$; consequently we obtain the (sub-)Stirling permutation $\sigma'' = 1122$, which has block decomposition $\sigma'' = [11][22]$. Hence, $X_{9,4}^{[B]}(\sigma) = 1$, $X_{9,3}^{[B]}(\sigma) = 1$ and $X_{9,2}^{[B]}(\sigma) = 1$.

Theorem 7. *For $j > 2$ the number of nodes $X_{n+1,j}$ of outdegree j in a random plane increasing tree of size $n+1$ coincides with $X_{n,j-1}^{[LR]}$, counting the number of size $j-1$ left-right trees in a random ternary increasing tree of size n, and with $X_{n,j}^{[B]}$ counting the number of sub-Stirling permutations with number of blocks equal to j, starting with a random Stirling permutation of size n,*

$$X_{n+1,j} = X_{n,j-1}^{[LR]} = X_{n,j}^{[B]}.$$

Moreover, the nodes of outdegree two in plane increasing tree of size $n+1$ correspond to the number of nodes in ternary increasing trees of size n having exactly one children, connected by a center edge, where this child is a leaf node.

The proof of the the result consists of a simple application of the bijection stated in [15], and is therefore omitted.

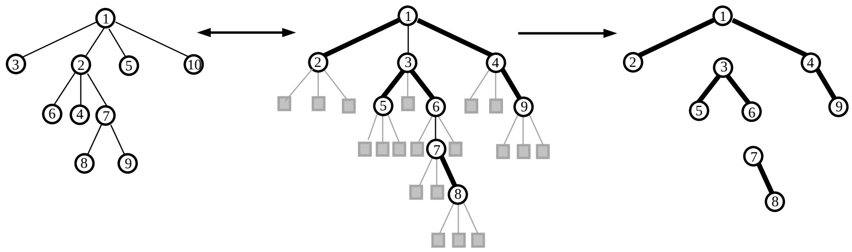

Figure 5: A plane-oriented increasing tree of size 10, the corresponding size 9 ternary increasing trees together with its left-right tree decomposition.

Example 9. The Stirling permutation σ of size nine corresponding to the trees in Figure 5, obtained either using the bijection with plane increasing tree [14] or with ternary increasing trees [15], is given by $\sigma = 221553367788614499$. As observed before we have $X_{9,4}^{[B]}(\sigma) = 1$, $X_{9,3}^{[B]}(\sigma) = 1$ and $X_{9,2}^{[B]}(\sigma) = 1$, corresponding to the number of nodes with outdegrees given by four, three and two in the corresponding plane increasing trees, and with the sizes of the left-right trees in ternary increasing trees.

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